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TEMPERATURE-COMPENSATED CUTS FOR VIBRATING BEAM RESONATORS OF GALLIUM ORTHOPHOSPHATE $GaPO_4$

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Abstract - **A theoretical investigation of rectangular cross-section $GaPO_4$ vibrating beam resonators is proposed. Flexural modes are the basic vibrating mode of tuning forks used in quartz wrist watches, and can also be used as sensors. Very little work, if any, has been done for vibrating beam resonators in $GaPO_4$.**

The goal is then to investigate the possibility of temperature-compensated cuts for all three kinds of vibrations in $GaPO_4$: extensional, flexural, and torsional modes by analytical methods. Modeling temperature effects is achieved by the approximate but classical method of varying effective elastic constants, beam dimensions and crystal mass density versus temperature.

Temperature-compensated cuts are found in $GaPO_4$ for length extensional modes and flexural modes. For vibrating beams, some of temperature-compensated cuts of $GaPO_4$ exhibit inversion points at high temperatures.

Keywords **- $GaPO_4$, vibrating beam, temperature-compensated cuts, extensional, flexural, and torsional modes.**

I. INTRODUCTION

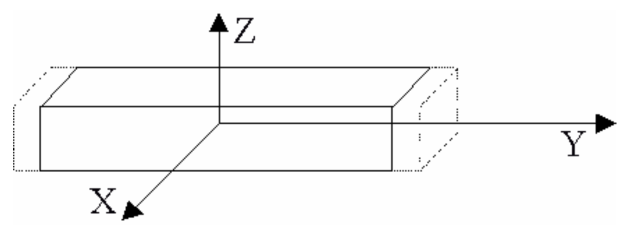
$GaPO_4$ is new quartz-homeotypic piezoelectric crystal. Its excellent thermal stability up to 970°C opens new fields of applications for high temperature sensors. Extensive studies on rotated Y-cut thickness shear resonators have been performed [1][2][3]. However, extensional and flexural vibrating beams of $GaPO_4$ used as resonators have never been reported yet. Vibrating modes could be used in $GaPO_4$ for physical sensors like vibrating beam accelerometers, pressure or temperature sensors.

Modeling the frequency dependence vs temperature is a pre-requisite for any kind of physical sensors thus it is necessary to find, if they exist, temperature-compensated cuts.

This paper reports on analytical models of vibrating beam resonators with a rectangular cross-section in extensional modes and flexural modes. Temperature effects are modeled by an approximate but very classical method of varying elastic constants, beam dimensions and crystal mass density versus temperature. It is shown that temperature-compensated cuts for vibrating beam resonators in extensional modes and flexural modes exist for $GaPO_4$.

Comparison with quartz shows some advantages of $GaPO_4$ over quartz and particularly the existence of high temperature inversion points.

II. ANALYTICAL MODEL OF LENGTH EXTENSIONAL MODES



A. Resonant frequencies [4][5]

We determine resonant frequencies from the equation of motion (1) although the analytical model of length extensional is built without taking into account the piezoelectric effect as a first approach. Hypothesis used in this model are:

length >> width and length >> thickness.

$$c^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

with $$c^2 = \frac{1}{S_{22}\rho}$$

The analytical formulae, giving the values of resonant frequencies (2, 3), are the same for all materials.

For a fixed-free beam:

$$f_n = \frac{2n-1}{4y_o}c \qquad (2)$$

For a fixed-fixed or free-free beam:

$$f_n = \frac{n}{2y_o}c \qquad (3)$$

Different material properties as mass density, elastic constants and beam dimensions allow a simple comparison between $GaPO_4$ and quartz.

In this study, we use only the rotated X-cut defined in Fig. 1. This simple rotation is sufficient to find temperature compensated cuts for $GaPO_4$ and quartz. The angle of rotation is called θ.

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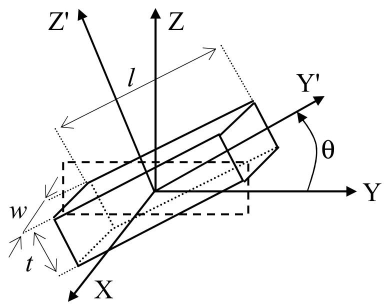


Fig. 1. Definition of rotated X-cut according to the crystallographic axes.

Fig. 2 represents a comparison of resonant frequencies of the first extensional mode for $GaPO_4$ and quartz. Frequencies are a function of the rotated X-cut. The curves have same form for both clamping conditions.

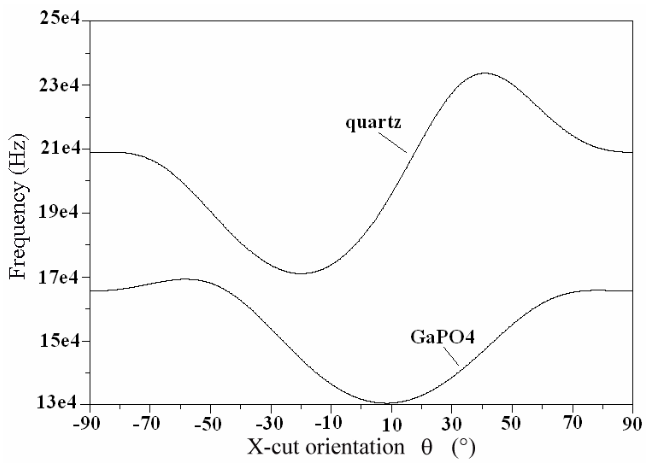


Fig. 2. Comparison between frequencies vs rotated X-cut for $GaPO_4$ and quartz for length extensional mode.
Beam dimensions: l = 15 mm, w = 2 mm, t = 1 mm.

B. Temperature effects

Using the approximate method which consists in varying elastic constants, beam dimensions and mass density versus temperature, after some calculus, we obtain an analytical expression of the first three Temperature Coefficients of Frequency (TCF) (4,5,6):

$$\alpha = \frac{1}{f}\frac{\partial f}{\partial T} = \frac{1}{2}\left(\frac{\dot{x}}{x} + \frac{\dot{z}}{z} - \frac{\dot{S}_{22}}{S_{22}} - \frac{\dot{y}}{y}\right) \tag{4}$$

$$\beta = \frac{1}{2!f}\frac{\partial^2 f}{\partial T^2} = \frac{1}{4}\left[\frac{\ddot{x}}{x} - \left(\frac{\dot{x}}{x}\right)^2 + \frac{\ddot{z}}{z} - \left(\frac{\dot{z}}{z}\right)^2 - \frac{\ddot{S}_{22}}{S_{22}} + \left(\frac{\dot{S}_{22}}{S_{22}}\right)^2 - \frac{\ddot{y}}{y} + \left(\frac{\dot{y}}{y}\right)^2\right] + \frac{1}{2}\alpha^2 \tag{5}$$

$$\gamma = \frac{1}{3!f}\frac{\partial^3 f}{\partial T^3} = \frac{1}{3!}\left[\left(log(f)\right)^{(3)} + 3\left(2!\beta\right) - 2\alpha^3\right] \tag{6}$$

Fig. 3 allows the comparison of the first TCF evolution for quartz and $GaPO_4$. Classical temperature-compensated cut is found again for quartz crystal at 7° (or 5° with Bechman coefficients). Temperature-compensated cuts exist for $GaPO_4$ in the vicinity of the X-cut. The two compensated angles for the first order are –10.2° and –50°. The second TCF β is the smallest for –10.2° then this cut is a better choice.

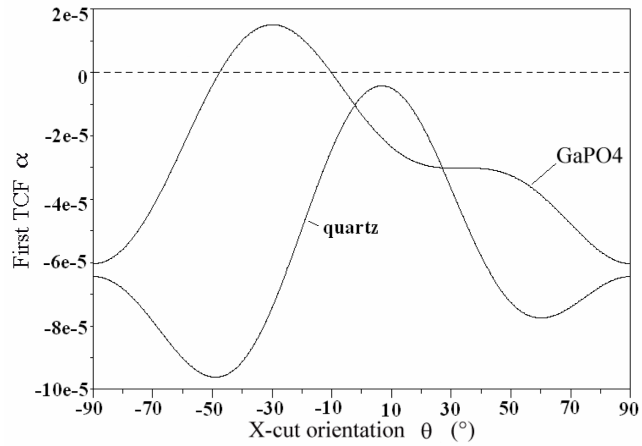


Fig. 3. Evolution of first TCF vs rotated X-cut for $GaPO_4$ and quartz for extensional mode.

For the temperature-compensated orientation, we show the different behaviour of the frequency temperature curve as compared between quartz and $GaPO_4$ (Fig. 4a and 4b).

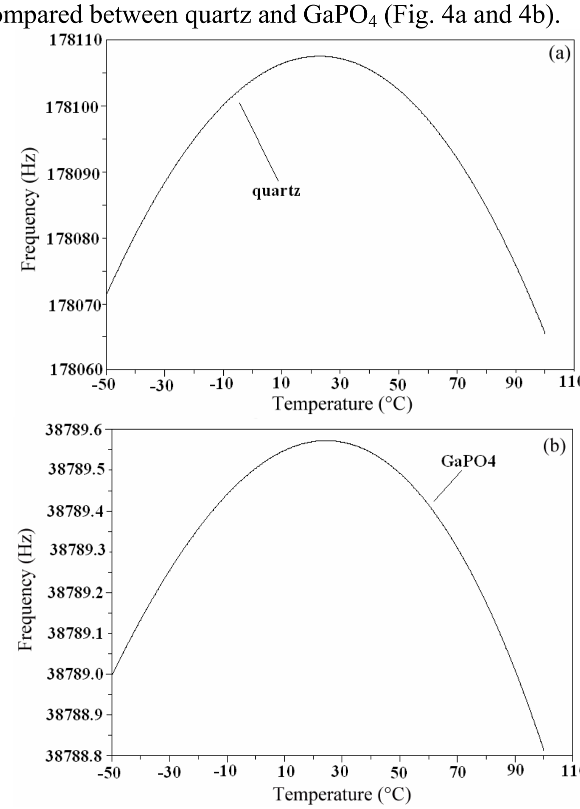
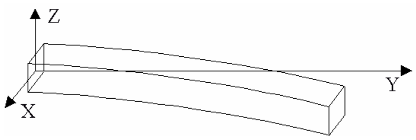


Fig. 4. Frequency vs temperature for temperature compensated cut.
(a) quartz: θ = 5°, (b) $GaPO_4$: θ = -10.2°.

Between –50 °C to 100 °C, the frequency variation for quartz is about 220 ppm whereas it is about 25 ppm for $GaPO_4$.

III. FLEXURAL MODES [4][5]



A. Resonant frequencies

In the same way, we study a vibrating beam resonator in a flexural mode. The derivation used in this model is based on the Bernoulli beam model where shear effects are neglected. This hypothesis is valid when the beam length is very superior to its width and thickness. For these hypothesis and this vibrating mode, the equation of motion (7) is classical:

$$Z^{IV} - \alpha^4 Z = 0 \qquad (7)$$

with $\alpha^4 = \frac{\rho S \omega^2}{EI}$ and $E = \frac{1}{S_{22}}$

And the resonant frequencies are:

$$f = \frac{\lambda^2}{2\pi y^2}\sqrt{\left(\frac{1}{S_{22}\rho S}\right)} \qquad (8)$$

where λ is solution of eigenfrequencies equation depending on boundary conditions. For a fixed-free beam the eigenfrequency equation is

$$1 + \cos(\lambda)ch(\lambda) = 0 \qquad (9)$$

and for a fixed-fixed beam

$$1 - \cos(\lambda)ch(\lambda) = 0 \qquad (10)$$

Fig. 5 gives the frequency variations versus rotated X-cut angle for the first flexural mode of the beam. We notice differences between $GaPO_4$ and quartz.

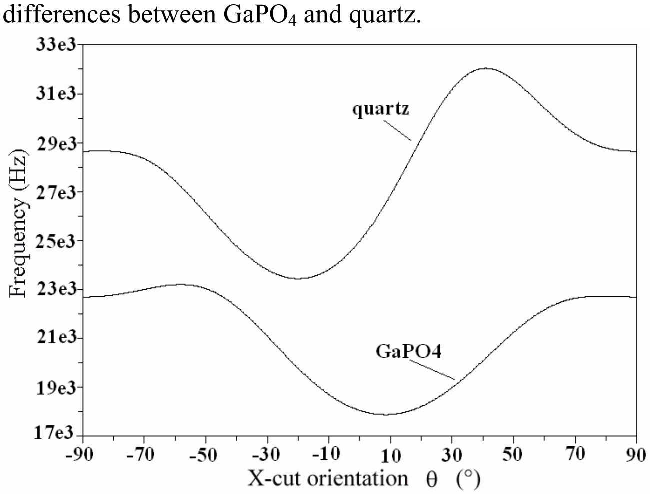


Fig. 5. Comparison between Frequencies vs rotated X-cut for $GaPO_4$ and quartz for flexural mode.
Beam dimensions: l = 15 mm, w = 2 mm, t = 1 mm.

B. Temperature effects

Using the approximate method, TCF α, β and γ are obtained by formal calculus [Maple]:

$$\alpha = \frac{1}{2}\left[\frac{\dot{x}}{x} + 3\frac{\dot{z}}{z} - 3\frac{\dot{y}}{y} - \frac{\dot{S}_{22}}{S_{22}}\right] \qquad (9)$$

$$\beta = \frac{1}{2}\left[\frac{1}{2}\left(\frac{\ddot{x}}{x} - \left(\frac{\dot{x}}{x}\right)^2 + 3\frac{\ddot{z}}{z} - 3\left(\frac{\dot{z}}{z}\right)^2 - 3\frac{\ddot{y}}{y} + 3\left(\frac{\dot{y}}{y}\right)^2 - \frac{\ddot{S}_{22}}{S_{22}} + \left(\frac{\dot{S}_{22}}{S_{22}}\right)^2\right) + \alpha^2\right] \qquad (10)$$

$$\gamma = \frac{1}{6}\left[\frac{1}{2}\left[\frac{x^{(3)}}{x} - 3\frac{\ddot{x}\dot{x}}{x^2} + 2\left(\frac{\dot{x}}{x}\right)^3 + \frac{z^{(3)}}{z} - 9\frac{\ddot{z}\dot{z}}{z^2} + 6\left(\frac{\dot{z}}{z}\right)^3 - 3\frac{y^{(3)}}{y} + 9\frac{\ddot{y}\dot{y}}{y^2} - 6\left(\frac{\dot{y}}{y}\right)^3 - \frac{S_{22}^{(3)}}{S_{22}} + 3\frac{\ddot{S}_{22}\dot{S}_{22}}{S_{22}^2} - 2\left(\frac{\dot{S}_{22}}{S_{22}}\right)^3\right] + 6\beta\alpha - 2\alpha^3\right] \qquad (11)$$

Fig. 6 reports the first TCF α for $GaPO_4$ and quartz. Some major differences appear between $GaPO_4$ and quartz.

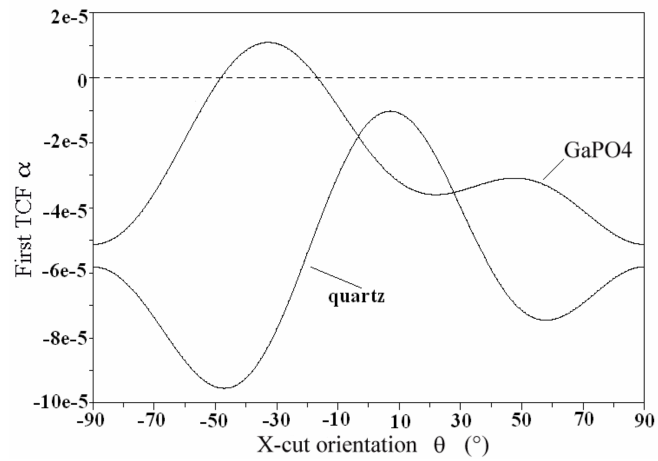


Fig. 6. Evolution of first TCF vs rotated X-cut for $GaPO_4$ and quartz for flexural mode.

The 1st TCF (9) for quartz is never equal to zero (Fig. 7). The TCF effects vs theta curve shows only a non zero minimum value about 1E-5 ppm.

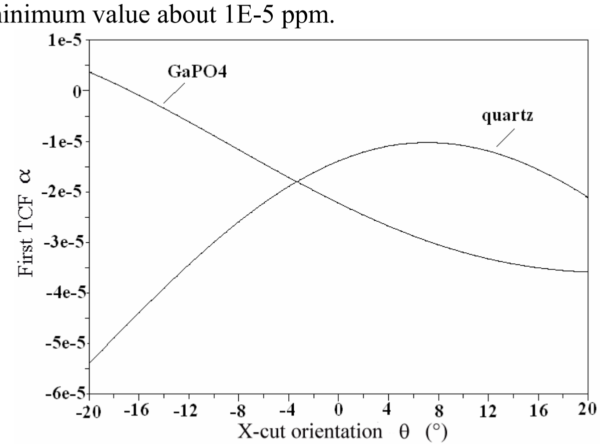


Fig. 7. Evolution of first TCF vs rotated X-cut for $GaPO_4$ and quartz for flexural mode between -20° and 20°.

Contrary to quartz, the 1st TCF (9) of $GaPO_4$ has a zero value for a particular angle θ = -16.7° (Fig. 6 and 7). Thus, compensation of temperature effects to the first order exists.

This characteristic gives some interesting advantages of $GaPO_4$ relative to quartz. Fig. 8a and 8b represent frequency for $GaPO_4$ and quartz versus temperature for their temperature-compensated cut, respectively. By slightly adjusting, the cut angle inversion points can be moved to high temperatures (Fig. 8c) for $GaPO_4$. This property being usable at high T, is a major advantage of $GaPO_4$ over quartz and justifies the interest of finding temperature compensated cuts at high temperature.

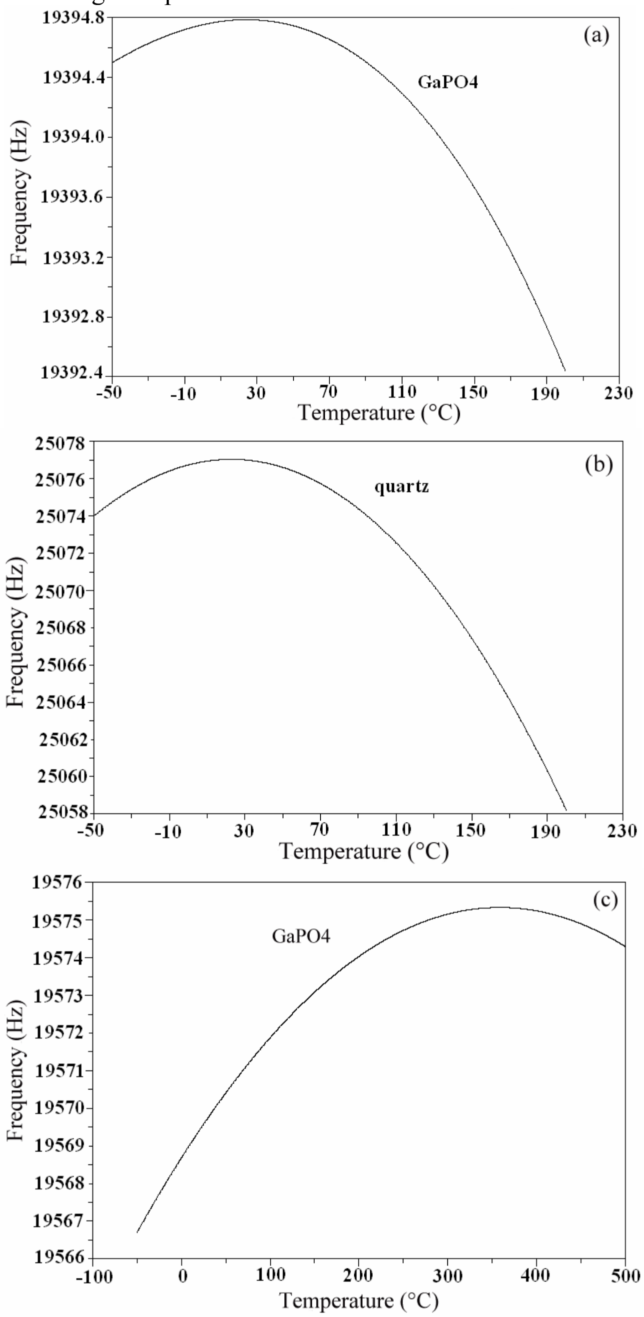
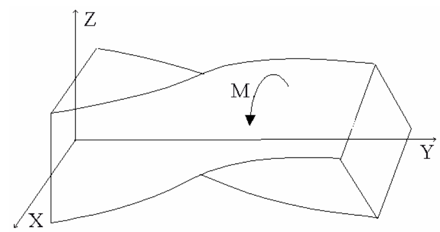


Fig. 8. Frequency vs temperature for temperature-compensated cut (a) Quartz, (b) $GaPO_4$, (c) $GaPO_4$ at high temperature $\theta = -18.2$ °.

For $GaPO_4$, temperature spread, where frequency is compensated, is more important than quartz. Between -50°C and 150°C, frequency variations are about 100 ppm for $GaPO_4$. For quartz, between the same temperature spread, frequency variations are more important and equal to about 800 ppm.

IV. TORSIONAL MODES [6][7][8][9]



A. Resonant frequencies

The hypothesis used in this model are based on Saint-Venant torsion problem. A derivation, not reproduced here, shows that the displacements can be derived from a potential $\phi(x,y)$ (12) for a rectangular cross-section.

$$\phi(x,z) = -xz + \frac{8z_o^2}{\pi^3}\sqrt{\left(\frac{C_{66}}{C_{44}}\right)} \ldots$$

$$\ldots \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \frac{\sinh\left(\frac{(2n+1)\pi x}{z_o}\sqrt{\left(\frac{C_{44}}{C_{66}}\right)}\right)}{\cosh\left(\frac{(2n+1)\pi x_o}{2z_o}\sqrt{\left(\frac{C_{44}}{C_{66}}\right)}\right)} \sin\left(\frac{(2n+1)\pi z}{z_o}\right) \quad (12)$$

The warping function (12) is drawn in Fig. 9 for a beam which has 2 mm width and 1 mm thickness.

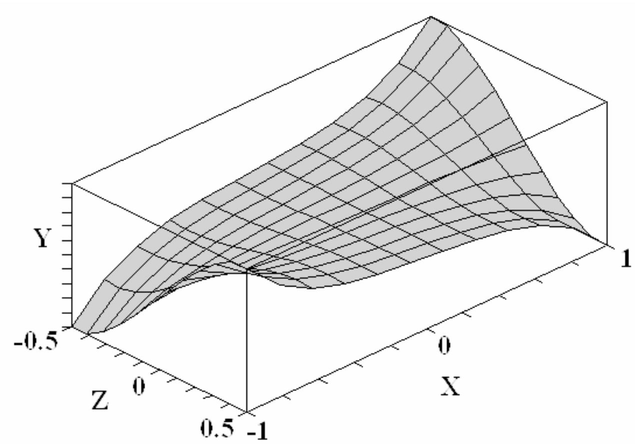


Fig. 9. Representation of warping section for vibrating beam resonator with a rectangular cross-section in torsional mode.

The resulting moment (13) is proportional to torsional angular τ (rd/m) and a coefficient C_t (14) depending of beam dimensions and elastic coefficients.

$$M = \tau C_t \quad (13)$$

$$C_t = \frac{C_{66}\, x_o z_o^3}{3}\left(1 - \frac{z_o}{x_o}\sqrt{\left(\frac{C_{66}}{C_{44}}\right)}\frac{192}{\pi^5} \ldots\right.$$

$$\left.\ldots \sum_{n=0}^{\infty} \frac{1}{(2n+1)^5\ \pi^5} \tanh\left(\frac{(2n+1)\pi x_o}{2z_o}\sqrt{\left(\frac{C_{44}}{C_{66}}\right)}\right)\right) \quad (14)$$

With these previous hypotheses, the equation of motion (15) and boundary conditions lead to resonant frequencies formulae (16, 17).

$$\rho I_y \frac{\partial^2 \alpha}{\partial t^2} - C_t \frac{\partial^2 \alpha}{\partial y^2} = 0 \qquad (15)$$

For a Fixed-free boundary conditions:

$$f_n = \frac{n}{4y_o}\sqrt{\left(\frac{C_t}{\rho I_y}\right)} \qquad (16)$$

For a Fixed-fixed boundary conditions:

$$f_n = \frac{n}{2y_o}\sqrt{\left(\frac{C_t}{\rho I_y}\right)} \qquad (17)$$

Fig. 10 represents frequency vs the rotated X-cut for $GaPO_4$ and quartz.

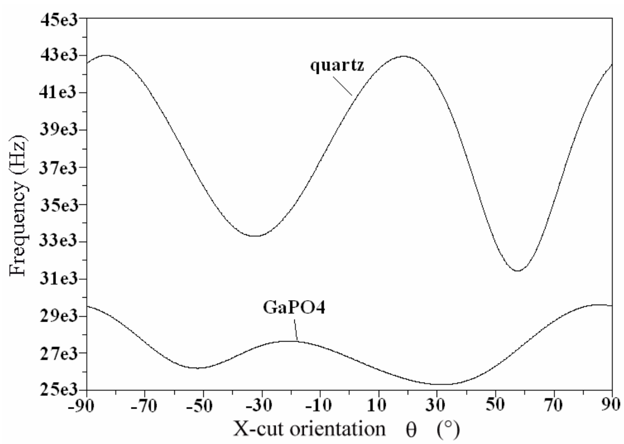


Fig. 10. Comparison between Frequencies vs rotated X-cut for $GaPO_4$ and quartz for the first torsional mode.
Beam dimensions: l = 15 mm, w = 2 mm, t = 1 mm.

For the 1st torsional mode, $GaPO_4$ resonant frequency values are always lower than quartz resonant frequencies. Moreover, the frequency variations are greater for quartz than $GaPO_4$. Consequently, excitation of a $GaPO_4$ resonators would be more easy.

V. Conclusion

An analytical model of rectangular cross-section $GaPO_4$ vibrating beam resonators has been proposed for extensional, flexural, and torsional modes. Modeling temperature effects is achieved for extensional and flexural modes.

Temperature-compensated cuts are found in $GaPO_4$ for length extensional modes (θ = -10.2°) and flexural modes (θ = -16.7°). Contrary to quartz, the 1st TCF of $GaPO_4$ has a zero value for particular angles. Thus, compensation of temperature effects exists.

Some of temperature-compensated cuts of $GaPO_4$ for vibrating beams exhibit inversion points at high temperatures and could be used for high temperature piezoelectric sensor applications.

Acknowledgment

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